Existence and Absence of Geometric Phases Due to Mode Transformations of High-Order Modes

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ABSTRACT

We study the geometric phase that is acquired when the second-order mode of an optical beam undergoes a cyclic transformation. We find that a geometric phase appears when the initial and intermediate modes have different quantities of orbital angular momentum. The phase is similar to the one measured previously for transformations of first-order modes. However, we find no accumulated geometric phase when the initial and intermediate modes have zero orbital angular momentum.

Keywords: geometric phase, high-order modes, orbital angular momentum

1. INTRODUCTION

Geometric phase or Berry phase is a topological phase that appears when a physical system undergoes a cyclic change in the space of parameters or states.[1] Since Berry's original discovery, geometric phases have been discovered in many physical contexts. What is interesting about this phase is that it is unbound, and in many cases unanticipated. In optics it has been found in many contexts: for cyclic paths in the direction of the spin of the photon (spin redirection phase),[2,3] for cyclic changes in the polarization state of the light (Pancharatnam phase),[4] and more recently for cyclic transformations in the high-order modes of a Gaussian beam (orbital phase).[5] The optical geometric phase has been found in many contexts, yielding many useful applications.[6] Because of the low dimension of the spaces studied thus far, the phase has always been represented geometrically in terms of paths in a spherical geometry. A geometric phase has been found in all cases.

The orbital phase offers the possibility of exploring higher dimensional spaces. Solutions to the paraxial wave equation give rise to well known Hermite-Gauss (HG) and Laguerre-Gauss (LG) modes.[7] These modes are represented in terms modes indices. In the rectangular HG_{nm} modes, n and m represent the number of nodes in the Hermite functions in x and y directions, respectively. In the cylindrical LG_p^ℓ modes, p represents the number of radial nodes in the Laguerre functions, and ℓ represents the multiples of 2π that the phase advances per full turn of the axial variable. Optical beams in the latter modes carry orbital angular momentum of $\ell\hbar$ per photon. Mode converters discovered thus far rephase the wavefront of the wave so as to produce transformations that preserve the order number $N = n + m = 2p + |\ell|$. In confined systems, such as a laser cavity, modes of different order have different energy. So far only the geometric phase in the space of order one has been studied.[5,8,9]

In addition, there is an aspect of the optical geometric phase that has been subject of debate: whether or not the geometric phase arises from the exchange of angular momentum between the light and the optical system.[10,11] In spin-redirection and Pancharatnam phase, the phase has been tied to the exchange of spin angular momentum, while the orbital phase involved an exchange of orbital angular momentum.[5] While this connection has been suggested, it has not been demonstrated. Van Enk's original conjecture[10] stated that if a geometric phase is mediated by angular momentum exchange, then a cyclic path between states of the same orbital angular momentum would not produce a geometric phase. In all previous studies the low dimensionality of the space forced exchanges of angular momentum for any closed topological path.

With the higher dimensional spaces of order N, the space of modes provides a potential system to test this conjecture. Orders greater than one may allow paths between states with the same value of the orbital angular

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momentum. For a given value of N all LG eigenmodes have a different value of ℓ. HG modes do not carry orbital angular momentum. Thus the only possibility for transformations between eigenmodes is for the case where $\ell = 0$. When N is odd, one of the LG solutions has $\ell = 0$, and thus can be used in conjunction with the HG modes for making a path where the initial, intermediate and final modes have no orbital angular momentum.

In this article we investigate the geometric phase in a higher dimensional space than previously considered. In particular we concentrate on the $N = 2$ mode space. We find results that are consistent with the angular momentum exchange conjecture. In the second section we present the theoretical context and in the third section we present the experimental set up for carrying the experimental tests.

2. MATHEMATICAL FRAMEWORK

In the space of first order modes ($N = 1$), suitably oriented $\pi/2$ converters transformed the phase front of the beam from a HG eigenstate to an LG eigenstate. This transformation, given by a unitary matrix operation on the HG eigenmode, had a corresponding path on the orbital Poincaré sphere.[9] Such a path was a geodesic that followed a meridian on the sphere from the south pole (LG_0^{-1}) to the Equator (HG_{01}). A closed path made of similar geodesics produced a geometric phase: one geodesic along the equator to a point of longitude 2θ (HG_{01} rotated by θ) and another geodesic one back from the equator to the pole. The geometric phase corresponded to the area enclosed by the curve on the sphere.[5] Varying the longitude of the second intermediate state changed the area and thus the geometric phase.

Analytically, the paths were represented by unitary 2×2 matrices and the modes by spinor vectors. The matrix T representing the closed path had as eigenvectors the initial/final modes ψ and as eigenvalue the geometric-phase (ϕ) shifting term:

$$T\psi = e^{i\phi}\psi, \tag{1}$$

In the space of second-order modes we no longer have the visual convenience of the orbital Poincaré sphere, but we can proceed with the analytical ray-tracing method using the space of first-order modes as a guide. Although it has yet to be demonstrated, we speculate that certain transformations can be reduced to subspaces that can be described by a spherical geometry.

We follow the matrix formalism of O'Neill and Courtial.[12] In the HG representation of order $N = 2$ the fundamental modes HG_{20}, HG_{11} and HG_{02} form the basis vectors

$$HG_{20} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad HG_{11} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad HG_{02} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{2}$$

Cyclic mode transformations involve the fundamental LG modes, which in the HG basis are given by

$$LG_0^{+2} = \frac{1}{2}\begin{pmatrix} -1 \\ i\sqrt{2} \\ 1 \end{pmatrix}, \quad LG_1^{0} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad LG_0^{-2} = \frac{1}{2}\begin{pmatrix} -1 \\ -i\sqrt{2} \\ 1 \end{pmatrix}. \tag{3}$$

In this work we will consider only modes and transformations that we can reproduce in the laboratory. For mode transformers we use the $\pi/2$ mode converter that transforms HG modes into LG modes and *vice versa*. We also use optical rotators to rotate the rectangular HG modes. The matrices representing them are given by

$$C_{\pi/2} = \begin{pmatrix} -i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{pmatrix}, \quad \text{and} \quad R(\alpha) = \begin{pmatrix} \cos^2\alpha & \frac{\sin 2\alpha}{\sqrt{2}} & \sin^2\alpha \\ -\frac{\sin 2\alpha}{\sqrt{2}} & \cos 2\alpha & \frac{\sin 2\alpha}{\sqrt{2}} \\ \sin^2\alpha & -\frac{\sin 2\alpha}{\sqrt{2}} & \cos^2\alpha \end{pmatrix}. \tag{4}$$

Some of the intermediate modes are rotated versions of the fundamental HG modes. For example, mode HG_{nm} rotated clockwise by an angle α is represented mathematically as:

$$HG_{nm}(\alpha) = R(\alpha) HG_{nm}. \tag{5}$$

One can verify, for example that $HG_{02} = R(\pi/2) HG_{20}$. In order to properly convert the rotated HG modes to LG modes we need a rotated $\pi/2$-converter. Analytically we obtain the matrix for the $\pi/2$-converter rotated by an angle α using

$$C_{\pi/2}(\alpha) = R(-\alpha) C_{\pi/2} R(\alpha). \tag{6}$$

We can pick a first geometric-phase path in analogy to the first-order modes:

$$T_1 : LG_0^{-2} \overset{C_{\pi/2}}{\rightarrow} HG_{02} \overset{R}{\rightarrow} HG_{02}(\theta) \overset{C_{\pi/2}}{\rightarrow} LG_0^{-2}. \tag{7}$$

Analytically, this expression is given by

$$T_1(\theta) = C_{\pi/2}(\pi/4 - \theta) R(\theta) C_{\pi/2}(-\pi/4) \tag{8}$$

In order to generate a geometric phase we need

$$T_1(\theta) LG_0^{-2} = e^{i\phi} LG_0^{-2}. \tag{9}$$

The relation between ϕ and θ in general depends on the geometry of the path, and as we will find here, on the initial/final mode. For the particular case of Eq. 9, we find $\phi = 2\theta$.

In the LG basis, where LG_0^{-2} is a basis vector, $T_1(\theta)$ should be diagonal. The matrices to change basis from HG to LG and *vice versa* are given by[12]

$$B_{HG \rightarrow LG} = \frac{1}{2} \begin{pmatrix} 1 & -i\sqrt{2} & -1 \\ \sqrt{2} & 0 & \sqrt{2} \\ 1 & i\sqrt{2} & -1 \end{pmatrix}, \qquad B_{LG \rightarrow HG} = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{2} & 1 \\ i\sqrt{2} & 0 & -i\sqrt{2} \\ -1 & \sqrt{2} & -1 \end{pmatrix} \tag{10}$$

Under this basis the fundamental LG modes are given by

$$LG_0^{+2} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad LG_1^0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad LG_0^{-2} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{11}$$

The matrix for the closed path T_1 is then given by

$$T_{1-LG} = B_{HG \rightarrow LG} T_1(\theta) B_{LG \rightarrow HG} = \begin{pmatrix} e^{i2\theta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i2\theta} \end{pmatrix}, \tag{12}$$

which is indeed diagonal. Notice that the same transformation gives a different geometric phase for different eigenmodes. In particular, LG_1^0 acquires no geometric phase. Under T_1 this mode transforms as:

$$T_1 : LG_1^0 \overset{C_{\pi/2}}{\rightarrow} HG_{11} \overset{R}{\rightarrow} HG_{11}(\theta) \overset{C_{\pi/2}}{\rightarrow} LG_1^0 \tag{13}$$

Notice that none of these states carry orbital angular momentum.

Consider now another geometric-phase path:

$$T_2 : HG_{20} \overset{C_{\pi/2}}{\rightarrow} LG_0^{+2} \overset{C_{\pi/2}}{\rightarrow} HG_{20}(\theta) \overset{R}{\rightarrow} HG_{20}. \tag{14}$$

This transformation is given by

$$T_2(\theta) = R(-\theta) C_{\pi/2}(-\pi/4 - \theta) C_{\pi/2}(\pi/4) \tag{15}$$

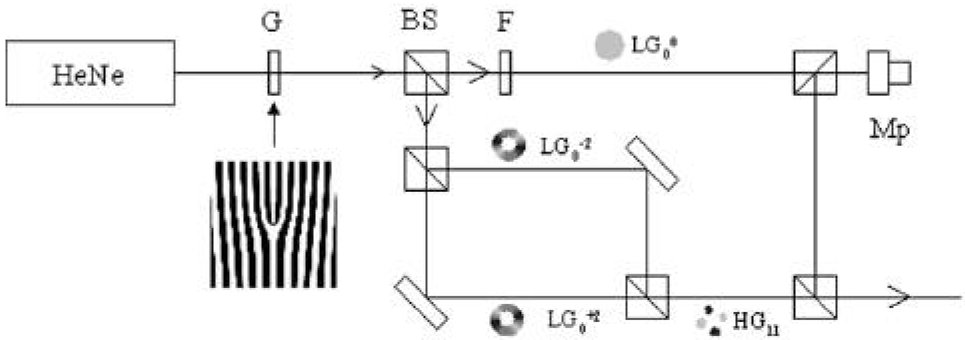

Figure 1. Schematic of the apparatus to produce second-order modes. A HeNe laser beam is used to generate $LG_0^{\pm 2}$ modes via a charge-2 forked grating (G). The modes were combined in two nested Mach-Zehnder interferometers, and interfered with a zero-order mode. The phase of the latter is changed with a piezo-driven mirror (Mp). Other elements include non-polarizing beam splitters (BS) and a neutral density filter to attenuate the reference beam.

This transformation indeed produces a geometric phase:

$$T_2(\theta)HG_{20} = e^{2i\theta}HG_{20} \tag{16}$$

Similarly to Eq. 12, we find that this transformation in the HG basis is diagonal:

$$T_2 = \begin{pmatrix} e^{i2\theta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i2\theta} \end{pmatrix}. \tag{17}$$

It also shows that the transformation of HG_{11}, which involves the path

$$T_2 : HG_{11} \stackrel{C_{\pi/2}}{\rightarrow} LG_1^0 \stackrel{C_{\pi/2}}{\rightarrow} HG_{11}(\theta) \stackrel{R}{\rightarrow} HG_{11}, \tag{18}$$

involves states bearing no orbital angular momentum, and produces no geometric phase.

In comparing T_1 and T_2 we find that the common feature of the two is that transformations involving orbital-angular-momentum exchanges (*e.g.,* $\Delta\ell = 2$ for $HG_{20} \rightarrow LG_0^{+2}$) produce non-zero geometric phases. The only paths that produce no geometric phase involve no net angular momentum exchanges between the initial and final modes in each of the transformations.

A generalization of this approach to mode spaces of order N can be subdivided in terms of the parity on N. For N odd all topological paths will generate a non-zero geometric phases. When N is even there will be sets of $\Delta\ell = 0$ transformations between $\ell = 0$ modes that do not generate a geometric phase. It also seems that the value of the geometric phase is limited by the maximum value of $\Delta\ell$ in the intermediate transformations of a given closed path. For example, for $N = 4$ there will be three possible geometric phases for a given path similar to the ones described here: 4θ, 2θ and 0.

3. EXPERIMENTAL METHOD

We used a method similar to the one used previously.[5] The apparatus consists of two parts. One part, shown in Fig 1, prepares the beam in the high-order mode under study and combines it with a reference beam. A second part, shown in Fig. 2, provides the mode transformations on the input beam.

The method uses interferometry with a collinear reference beam, a method first used by us in the analysis of modes.[5] This method is similar to the one used previously in studies of the Pancharatnam phase.[13] The incident beam in the high-order mode is combined with a reference beam in the zero-order mode (*i.e.,* $HG_{00} = LG_0^0$) and

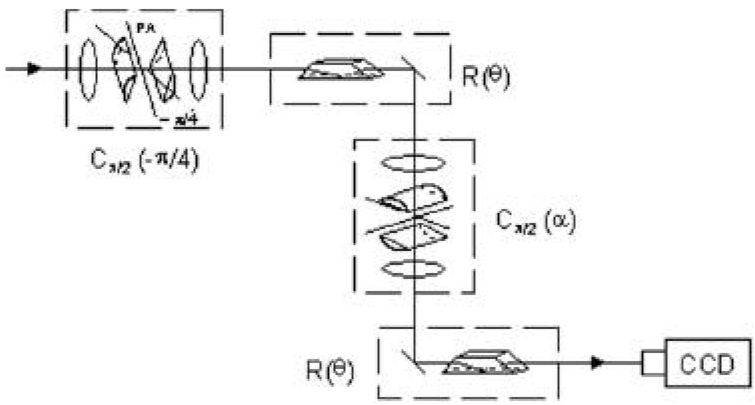

Figure 2. Schematic of the sequence of transformers used for measuring geometric phases for transformations T_1 and T_2 (see text).

sent collinearly through all the transformers. Since the $N = 0$ mode is the only one in its space it does not get transformed at all and thus acquires no geometric phase. However, by sending it collinearly with the high-order mode we subtract out dynamical phases produced by the apparatus when the geometric phase is varied.

As shown in the schematic of Fig. 1, the zero-order output of a 5 mW HeNe laser beam was sent to a charge-2 forked diffraction grating. This was an amplitude binary grating that we fabricated by photoreduction of a computer-generated pattern.[14] The central (LG_0^0) and two first-order diffracted beams, in the modes LG_0^{+2} and LG_0^{-2}, were sent to two nested Mach-Zehnder interferometers. The central LG_0^0 mode, used as a reference beam, took the top branch of external Mach-Zehnder interferometer. It was reflected by a mirror that was translated by piezo electric transducers. We used this to add a dynamical phase between the reference beam and the beam in the high-order mode.

Each of the modes LG_0^{+2} and LG_0^{-2} was sent through one the arms of the internal Mach-Zehnder interferometer. For the cyclic mode transformation T_1, we used one of the modes LG_0^{+2} or LG_0^{-2} by blocking the proper arm of the interferometer. For transformation T_2 we prepared mode HG_{11} by interfering the LG_0^{+2} and LG_0^{-2} modes with the appropriate phase.

The output of the Mach-Zehnder interferometer was sent to the sequence of mode transformers shown in Fig 2. The first transformer was a fixed $\pi/2$-converter, and the third transformer was a rotatable $\pi/2$-converter. The second and fourth transformers were rotators. Each rotator consisted of a mirror and a rotatable Dove prism. The combination produced an image rotation without inversion. Transformation T_1 was made using the first three transformers, and keeping the last rotator fixed at $\theta = 0$. For transformation T_2 we used the first, third and fourth transformers, keeping the second rotator fixed at $\theta = 0$.

To test the input modes we imaged them before and after the first mode converter. Figure 3a shows the predicted transformation of the modes. The different shades of gray represent the differing phases of the mode of the light on a transverse plane. For the case of LG_0^{+2} (first column) the input mode has the characteristic doughnut-shaped profile with a phase the winds by 4π in one turn. After the converter we get a rotated $HG_{20}(\pi/4)$ mode, which has three lobes, with the middle one out of phase with the outer two by π. In the second column we have as input the rotated $HG_{11}(\pi/4)$ mode, which gets converted into the LG_1^0 mode by the $\pi/2$-converter. The lobes of $HG_{11}(\pi/4)$ that are opposite to each other are in phase, but are out of phase by π with the other two. The LG_1^0 mode has a center spot surrounded by a ring with opposite phase. Shown in row (b) of Fig. 3 are the images of the modes before and after the conversion. The images were taken in false color and then converted to gray scale. All images in row (b) show the expected patterns. The last two rows (c) and (d) show the images that result when we allow the reference beam to go through so that it interfered with the mode. The phase of the reference beam was adjusted so that in (d) it was close to π relative to that in row (c). We can see that the regions of constructive interference shifter accordingly: for LG_0^{+2} they rotated by

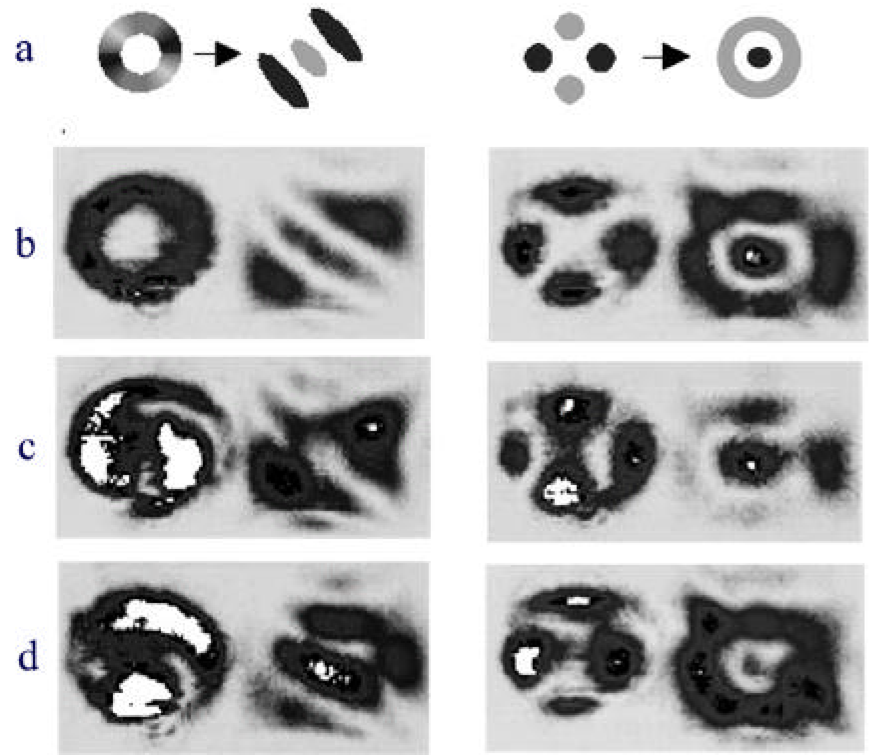

Figure 3. Modes before and after converter. In the first and second columns we have the transformations $LG_0^{+2} \rightarrow HG_{20}(\pi/4)$ and $HG_{11}(\pi/4) \rightarrow LG_1^0$, respectively. Row (a) is the predicted transformation. Row (b) is the measured images of the modes. Rows (c) and (d) are the interference of the modes with the zero-order reference beam in a different phase relative to the high-order modes.

90 degrees; for $HG_{20}(\pi/4)$ they switched from the outside lobes being intense to the middle; for $HG_{11}(\pi/4)$ the diagonal lobes switched in intensity; and for LG_1^0 the center spot switched in brightness with the surrounding ring. Deviations from the expected (ideal) intensity distributions were caused by the smaller size and slight asymmetry of the reference beam. The latter was caused by the amplitude grating that we used.

4. PRELIMINARY RESULTS AND CONCLUSIONS

Our preliminary results confirm the predictions presented here. When using state LG_0^{-2} and transformation T_1, which result in exchanges of orbital angular momentum between the light and the optical system, we observe a geometric phase. Although our experiments are ongoing, preliminary results indicate that the geometric phase is indeed 2θ. In addition, when we use HG_{11} as the initial state with transformation T_2, we observe no phase shift in the light as we change the topology of the path (*i.e.*, θ).

In summary, we have explored a new higher-dimensional regime of the orbital geometric phase in a space of modes that had not been studied before. We find that the geometric phases involved depend on the states involved as well as the paths. While we were not able to find a path that involved a constant value of ℓ, the absence of geometric phase for transformations involving $\ell = 0$ states, regardless of its topology, supports the conjecture that ties the geometric phase to the exchange of angular momentum between the light and the optical system.

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