Deducing 3-Dimensional Polarization Fields from Projective Measurements

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ABSTRACT

We developed a method to extract information of 3-dimensional coherent fields using projective measurements. We are particularly interested in 3-dimensional fields that are produced by the intersection of two orthogonally polarized light beams. We find that for specific symmetric conditions we are able to deduce the 3-dimensional polarization ellipse of the every point in space.

Keywords: Polarization, Singular Optics

1. INTRODUCTION

In recent years there have been interesting predictions about 3-dimensional (3-d) electromagnetic fields. These include the existence of Möbius strips and twisted ribbons in the intersection of singular optical beams,[1–5] but also in the focusing of Poincaré beams.[6] There are also interesting discussions on the fields that appear upon the reflection at surfaces.[7] Thus, it would be interesting to confirm these predictions experimentally.

One method to determine 3-d fields involves the scattering of focused fields by a metallic microsphere.[8] A mapping of the scattered fields can be used to determine the field at the location of the microsphere.[8] We have been investigating obtaining 3-d fields using a simpler method: using polarization projections.[9] This, however, has its challenges: polarization projection cannot distinguish the longitudinal component of the fields from the component coplanar with the longitudinal component. In this article we find that under certain conditions we can determine 3-d fields. We present the method to extract the 3-d fields, to obtain the polarization at every location.

2. MODELING THE FIELDS

This article uses the same formalism as a previous SPIE Proceedings article on the subject.[9] We prepare the optical beams in two spatial modes. The two beams form an angle θ with the normal to the observation plane, as shown in Fig. 1. The equations for the field components of each mode in the observation plane are:

$$\vec{E}_\ell = E_\ell \left(\cos\theta \hat{e}_x - i\hat{e}_y + \sin\theta \hat{e}_z\right) \tag{1}$$

$$\vec{E}_0 = E_0 \left(\cos\theta \hat{e}_x + i\hat{e}_y - \sin\theta \hat{e}_z\right), \tag{2}$$

where E_ℓ and E_0 are complex. The total field is then

$$\vec{E} = (E_\ell + E_0)\cos\theta \hat{e}_x + (-E_\ell + E_0)i\hat{e}_y + (E_\ell - E_0)\sin\theta \hat{e}_z. \tag{3}$$

We can express the field as

$$\vec{E} = E_x \hat{e}_x + E_y \hat{e}_y + E_z \hat{e}_z \tag{4}$$

$$= |E_x|e^{i\delta_x}\ \hat{e}_x + |E_y|e^{i\delta_y}\ \hat{e}_y + |E_z|e^{i\delta_z}\ \hat{e}_z. \tag{5}$$
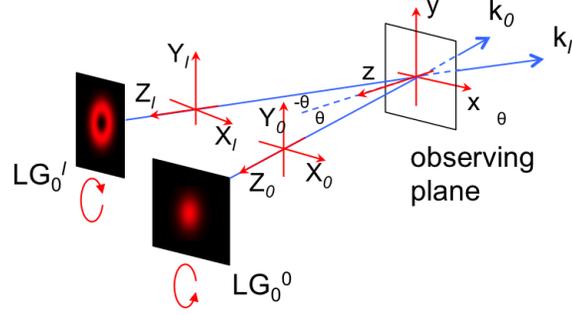
Figure 1. Reference frames of the two non-collinear beams and the observing frame.

We can rewrite the field components as

$$\vec{E} = e^{i\delta_x} \left(|E_\ell + E_0| \cos\theta \, \hat{e}_x + |E_\ell - E_0| e^{i(\delta - \pi/2)} \hat{e}_y + |E_\ell - E_0| \sin\theta \, e^{i\delta} \hat{e}_z \right), \tag{6}$$

where

$$\delta = \delta_y - \delta_x. \tag{7}$$

The term within parenthesis in Eq. 6 is the field to within an overall phase. The geometry of our problem makes the magnitudes of E_z proportional to that of E_y. By choice of polarization, the two components are also exactly $\pi/2$ out of phase. Neglecting constants, the intensity is

$$I = |E_x|^2 + |E_y|^2 + |E_z|^2, \tag{8}$$

In the experiment we use thin polarizers to project the state of the light. When a beam with propagation direction $\hat{k}$ and electric field $\vec{E}$ is incident on a polarizer with transmission axis along vector $\hat{p}$, the transmitted intensity is given by

$$I_p = \left| (\hat{e}_p \cdot \vec{E}) \hat{e}_p \right|^2, \tag{9}$$

where[10]

$$\hat{e}_p = \frac{\hat{p} - \hat{k}(\hat{k} \cdot \hat{p})}{\sqrt{1 - (\hat{k} \cdot \hat{p})^2}} \tag{10}$$

is the unit vector orthogonal to the propagation direction but in a plane that contains the polarizer transmission direction $\hat{p}$. Note that this leads to the conclusion that for a single beam forming an angle θ with the normal to the polarizer, the transmitted intensity is the sum of the squares of the $x-$ and $z-$components. Thus, a polarizer is unable to distinguish between them. This does not hinder our efforts because we can infer the $x-$ and $z-$components from the knowledge of θ and the measured intensity.

Applying this projective approach to the vertical direction ($\hat{p} = \hat{e}_y$), yields $\hat{e}_p = \hat{e}_y$ using

$$\hat{k}_\ell = \sin\theta \, \hat{e}_x - \cos\theta \, \hat{e}_z, \tag{11}$$
$$\hat{k}_0 = -\sin\theta \, \hat{e}_x - \cos\theta \, \hat{e}_z. \tag{12}$$

This leads to

$$I_V = \left| \vec{E}_\ell \cdot \hat{e}_y + \vec{E}_0 \cdot \hat{e}_y \right|^2, \tag{13}$$
$$= |-i(E_\ell - E_0)|^2 \tag{14}$$
$$= |E_y|^2. \tag{15}$$

Note that if we know θ, we can get the $z-$component of the field from Eqs. 6 and 15:

$$|E_z|^2 = I_V \sin^2\theta. \tag{16}$$

For the horizontal direction, we apply

$$I_H = \left| (\vec{E}_\ell \cdot \hat{e}_{x,\ell})\hat{e}_{x,\ell} + (\vec{E}_0 \cdot \hat{e}_{x,0})\hat{e}_{x,0} \right|^2, \tag{17}$$

where

$$\hat{e}_{x,\ell} = \cos\theta\,\hat{e}_x + \sin\theta\,\hat{e}_z \tag{18}$$
$$\hat{e}_{x,0} = \cos\theta\,\hat{e}_x - \sin\theta\,\hat{e}_z, \tag{19}$$

yielding

$$I_H = |E_\ell + E_0|^2 \cos^2\theta + |E_\ell - E_0|^2 \sin^2\theta. \tag{20}$$

From Eqs. 6 and 20 we get

$$|E_x|^2 = I_H - I_V \sin^2\theta. \tag{21}$$

We note that this simplification is possible only because both beams' propagation direction form the same angle (θ) with the normal to the observing plane. Thus, from the measurements of the intensities and knowledge of θ we can extract the amplitude squared of the field components. This is not enough to obtain the total field because we do not know the phase difference between the field components δ. As is common in polarimetry, we find these phase from other projective measurements. The projective measurements along the diagonal direction with $\hat{p} = \hat{e}_D = 2^{-1/2}(\hat{e}_x + \hat{e}_y)$ can be accounted by

$$I_D = \left| (\vec{E}_\ell \cdot \hat{e}_{D,\ell})\hat{e}_{D,\ell} + (\vec{E}_0 \cdot \hat{e}_{D,0})\hat{e}_{D,0} \right|^2, \tag{22}$$

with

$$\hat{e}_{D,\ell} = \frac{\cos^2\theta\,\hat{e}_x + \hat{e}_y + \sin\theta\cos\theta\,\hat{e}_z}{\sqrt{2 - \sin^2\theta}} \tag{23}$$

$$\hat{e}_{D,0} = \frac{\cos^2\theta\,\hat{e}_x + \hat{e}_y - \sin\theta\cos\theta\,\hat{e}_z}{\sqrt{2 - \sin^2\theta}} \tag{24}$$

Replacing Eqs. 1, 2, 23 and 24 into 22 yields

$$I_D = \frac{|E_x + E_y|^2 (\cos^4\theta + 1) + |E_y\cos^2\theta - E_x|^2 \sin^2\theta}{(2 - \sin^2\theta)^2}. \tag{25}$$

Similarly, along the antidiagonal direction, with $\hat{p} = \hat{e}_A = 2^{-1/2}(\hat{e}_x - \hat{e}_y)$, and

$$\hat{e}_{A,\ell} = \frac{\cos^2\theta\,\hat{e}_x - \hat{e}_y + \sin\theta\cos\theta\,\hat{e}_z}{\sqrt{2 - \sin^2\theta}} \tag{26}$$

$$\hat{e}_{A,0} = \frac{\cos^2\theta\,\hat{e}_x - \hat{e}_y - \sin\theta\cos\theta\,\hat{e}_z}{\sqrt{2 - \sin^2\theta}} \tag{27}$$

we get

$$I_A = \frac{|E_x - E_y|^2 (\cos^4\theta + 1) + |E_y\cos^2\theta + E_x|^2 \sin^2\theta}{(2 - \sin^2\theta)^2}. \tag{28}$$

The difference of the two intensities gives

$$I_D - I_A = 4|E_x||E_y|\sin\delta\,\frac{2\cos^4\theta - \cos^2\theta + 1}{(2 - \sin^2\theta)^2}, \tag{29}$$
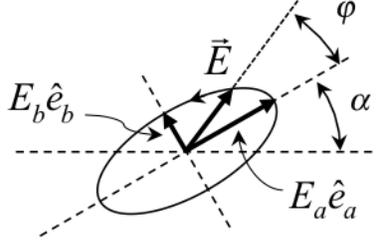
from which δ can be obtained.

Figure 2. Polarization ellipse, where the rectification phase γ defines the instantaneous angle φ that the electric field forms with the semi-major axis of the ellipse.

The electric field of the light describes in general an ellipse. If we express the electric field in terms of the field components along the semi-major and semi-minor axis of the ellipse, then those field components are 90 degrees out of phase.[11] If the unit vectors along the semi-major and semi-minor axes are $\hat{e}_a$ and $\hat{e}_b$, respectively, then the electric field is given by[4,12]

$$\vec{E} = e^{-i\gamma} \left(E_a \hat{e}_a - i E_b \hat{e}_b \right), \tag{30}$$

where E_a and E_b are real magnitudes and γ is also known as the rectification phase.[13,14] The sign between the ellipse-axis vectors is consistent with the right circular polarization convention $\hat{e}_R = 2^{-1/2}(1 - i)$: a positive phase is a counter-clockwise rotation of the polarization vector when looking into the beam. The rectification phase is related to the angle φ that the instantaneous electric field vector forms relative to the semi-major axis, as shown in Fig. 2:

$$\tan\varphi = \frac{E_b}{E_a} \tan\gamma. \tag{31}$$

The vectors representing the semi-major and semi-minor axes of the ellipse are therefore given by

$$\vec{a} = E_a \hat{e}_a = \mathrm{Re}(\vec{E}^* e^{i\gamma}) \tag{32}$$

$$\vec{b} = E_b \hat{e}_b = \mathrm{Im}(\vec{E}^* e^{i\gamma}) \tag{33}$$

$$\tag{34}$$

The phase γ can be obtained from[4]

$$e^{i\gamma} = \frac{\sqrt{\vec{E} \cdot \vec{E}}}{\left| \sqrt{\vec{E} \cdot \vec{E}} \right|}. \tag{35}$$

The electric field that we can construct by our method is given by

$$\vec{E'} = |E_x|\, \hat{e}_x + |E_y| e^{i(\delta - \pi/2)} \hat{e}_y + |E_z|\, e^{i\delta} \hat{e}_z, \tag{36}$$

with $|E_x|$, $|E_y|$, $|E_z|$ and δ obtained by Eqs. 21, 15, 16 and 29, respectively. The field of Eq. 36 is equivalent to Eq. 6 to within an overall phase. We can then apply the expression of Eq. 35 to the field of Eq. 36, to find the rectification phase for this field, and obtain the semi axes of the ellipse per Eqs. 32 and 33. That is, since Eqs. 6 and 36 differ by an overall phase, they just specify different positions of the field vector along the polarization ellipse, as shown in Fig. 2.

3. SUMMARY AND CONCLUSIONS

In summary, we have presented a method to extract the polarization ellipse in 3-d fields. Our method has a couple of restrictions: the viewing plane must be symmetric with respect to the two beams, and it calculates the field of coherent beams. The first restriction simplifies our algebra due to cancellations. This is important for obtaining the phase δ. The second restriction is due to our use of only 4 polarization projections. That is, we find the polarization on the surface of the Poincaré sphere only. The addition of projections onto the circular states would give us states in the interior of the sphere. We have made successful comparisons between modelings and measurements, which will be presented elsewhere. Another subtle challenge is the determination of δ in Eq. 29:

we can only obtain it modulo π. This, however, is not a huge setback because it gets the direction of the ellipse semi-axis vectors to within a sign: either as, for example, $\vec{a}$ or $-\vec{a}$. This restriction does not prevent us from extracting the ellipticity and orientation of the polarization ellipse.

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